Example –

(1) Prove that $\underset{x \to a}{Lt} \frac{x^2 - a^2}{x - a}$ exists. Find its value. [DR. SUNIL JEE, S.B. COLLEGE]

Solution – Here, $f(x) = \frac{x^2 - a^2}{x - a}$

$f(x)$ would exist at $x = a$.

When,

$$\underset{h \to 0}{Lt} f(a+h) = \underset{h \to 0}{Lt} f(a-h).$$

$$\because \underset{h \to 0}{Lt} f(a+h) = \underset{h \to 0}{Lt} \frac{(a+h)^2 - a^2}{\cancel{a} + h - \cancel{a}}$$

$$= \underset{h \to 0}{Lt} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{a^2}}{h}$$

$$= \underset{h \to 0}{Lt} \frac{2ah + h^2}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\cancel{h}[2a + h]}{\cancel{h}}$$

$$= \underset{h \to 0}{Lt} (2a + h)$$

$$= 2a.$$

Again, $\underset{h \to 0}{Lt} f(a-h) = \underset{h \to 0}{Lt} \frac{(a-h)^2 - a^2}{\cancel{a} - h - \cancel{a}}$

$$= \underset{h \to 0}{Lt} \frac{\cancel{a^2} - 2ah + h^2 - \cancel{a^2}}{-h}$$

$$= \underset{h \to 0}{Lt} \frac{h^2 - 2ah}{-h}$$

$$= \underset{h \to 0}{Lt} \frac{\cancel{h}(h - 2a)}{-\cancel{h}}$$

$$= \underset{h \to 0}{Lt} (2a - h)$$

$$= 2a.$$

$\therefore$ Left limit = Right limit = $2a$

Hence the given limit exist and its value is $2a$.

M.U. 13(H) (2) If $f(x) = x \sin\frac{1}{x}$, for $x \neq 0$, and $f(0) = 0$, Show that $f(x)$ is Continuous at $x = 0$.

Solution: For $f(x)$ to be Continuous at $x=0$

$$\underset{h\to 0}{Lt}\, f(0+h) = f(0) = \underset{h\to 0}{Lt}\, f(0-h)$$

$$\underset{h\to 0}{Lt}\, f(0+h) = \underset{h\to 0}{Lt}\,(0+h)\sin\left(\frac{1}{0+h}\right)$$

$$= \underset{h\to 0}{Lt}\, h \cdot \sin\frac{1}{h}.$$

As we know that the value of $\sin\frac{1}{h}$ lies between -1 & 1.

$$= \underset{h\to 0}{Lt}\, h \text{ (the values lies bet}^{n}\ -1 \,\&\, 1)$$

$$= 0.$$

$$f(0) = 0$$

$$\underset{h\to 0}{Lt}\, f(0-h) = \underset{h\to 0}{Lt}\,(0-h)\sin\left(\frac{1}{0-h}\right)$$

$$= \underset{h\to 0}{Lt}\, -h \times \sin\left(-\frac{1}{h}\right)$$

$$= \underset{h\to 0}{Lt}\, \cancel{-}h \times \cancel{-}\sin\frac{1}{h}$$

As $(\sin(-\theta) = -\sin\theta$.

$$= \underset{h\to 0}{Lt}\, h \sin\left(\frac{1}{h}\right) = 0$$

as above

$\therefore$ Thus, we see that

$$\underset{h\to 0}{Lt}\, f(0+h) = f(0) = \underset{h\to 0}{Lt}\, f(0-h) = 0$$

Hence, $f(x)$ is Continuous at $x=0$.

Ex. 3 Prove that the function,

$$f(x) = \frac{x^2}{a} - a \quad \text{for } 0 < x < a.$$

$$f(x) = 0 \quad \text{for } x = a$$

$$f(x) = a - \frac{a^3}{x^2} \quad \text{for } x > a$$

is Continuous at $x = 0$.

Soln. For $f(x)$ to be Continous at $x=a$.

$$\lim_{h\to 0} f(a+h) = f(a) = \lim_{h\to 0} f(a-h).$$

$$\therefore \lim_{h\to 0} f(a+h) = \lim_{h\to 0} a - \frac{a^3}{(a+h)^2}$$

$$= a - \frac{\cancel{a^3}\, a}{\cancel{a^2}}$$

$$= a - a = 0.$$

$$f(a) = a.$$

$$\lim_{h\to 0} f(a-h) = \lim_{h\to 0} \frac{(a-h)^2}{a} - a$$

$$= \frac{a^2}{a} - a = a - a = 0.$$

$$\therefore \lim_{h\to 0} f(a+h) = f(a) = \lim_{h\to 0} f(a-h)$$

$\therefore f(x)$ is Continous at $x = a$.

Ex. 4 Prove that the function $f(x)$

$$f(x) = \frac{e^{1/x}}{1+e^{1/x}},\ f(0) = 0 \text{ is}$$

discontinuous at $x = 0$.

Soln. For $f(x)$ to be Continuous at $x=0$

$$\underset{h\to 0}{Lt}\ f(0+h) = f(0) = \underset{h\to 0}{Lt}\ f(0-h)$$

$$\therefore \underset{h\to 0}{Lt}\ f(0+h) = \underset{h\to 0}{Lt}\ \frac{e^{\frac{1}{0+h}}}{1+e^{\frac{1}{0+h}}}$$

$$= \underset{h\to 0}{Lt}\ \frac{e^{1/h}}{1+e^{1/h}}$$

Dividing by $e^{1/h}$ we have

$$= \underset{h\to 0}{Lt}\ \frac{1}{\frac{1}{e^{1/h}}+1}$$

$$= \frac{1}{\frac{1}{e^{1/0}}+1} = \frac{1}{\frac{1}{e^{\infty}}+1}$$

$$= \frac{1}{e^{-\infty}+1} \quad [\because e^{-\infty}=0]$$

$$= \frac{1}{0+1} = 1.$$

Now, $f(0-h) = \underset{h\to 0}{Lt}\ \frac{e^{\frac{1}{0-h}}}{1+e^{\frac{1}{0-h}}}$

$$= \underset{h\to 0}{Lt}\ \frac{e^{-1/h}}{1+e^{-1/h}}$$

$$= \frac{e^{-1/0}}{1+e^{-1/0}}$$

$$= \frac{e^{-\infty}}{1+e^{-\infty}} = \frac{0}{1+0} = \frac{0}{1} = 0.$$

$f(0) = 0.$

$$\therefore \underset{h\to 0}{Lt}\ f(0+h) \neq f(0) = \underset{h\to 0}{Lt}\ f(0-h).$$

Hence $f(x)$ is not continuous at $x=0$.